MATH2060B Midterm I Solution

1(a) Since f is differentiable at c, $\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}=f'(c)$ exists.

$\lim_{x\to c}(f(x)-f(c))=\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}\lim_{x\to c}(x-c)=f'(c)\cdot 0=0$

Hence f is continuous at c.

Alternative solution:

Since f is differentiable at c, $\exists\delta_0>0$ s.t. for $0<|x-c|<\delta_0$, $|\frac{f(x)-f(c)}{x-c}-f'(c)|<1$.

$\Rightarrow |f(x)-f(c)|-|f'(c)||x-c|\le|f(x)-f(c)-f'(c)(x-c)|<|x-c|$.

$\Rightarrow |f(x)-f(c)|<|x-c|(|f'(c)|+1)$.

Given $\epsilon>0$, let $\delta=\min\{\delta_0,\frac{\epsilon}{|f'(c)|+1}\}$.

Then for $0<|x-c|<\delta$, $|f(x)-f(c)|<\delta(|f'(c)|+1)\le\epsilon$.

1(b) Suppose $x<y$. By mean value theorem, $\exists c\in(x,y)$ s.t. $f(y)-f(x)=f'(c)(y-x)$.

Since f' is strictly decreasing, $f'(c)<f'(x)$ hence $f(y)-f(x)<f'(x)(y-x)$.

Suppose $x>y$. By mean value theorem, $\exists c\in(y,x)$ s.t. $f(y)-f(x)=f'(c)(y-x)$.

Since $f'(c)>f'(x)$ and $y-x<0$, we have $f(y)-f(x)<f'(x)(y-x)$.

Alternative solution:

Since f' is strictly decreasing, f is strictly concave.

Suppose $x<y$. Let $w\in(x,y)$.

For any $z<x$, $\dfrac{f(y)-f(x)}{y-x}<\dfrac{f(w)-f(x)}{w-x}<\dfrac{f(x)-f(z)}{x-z}$.

Taking limit $z\to x^-$, $\dfrac{f(y)-f(x)}{y-x}<\dfrac{f(w)-f(x)}{w-x}\le f'_-(x)=f'(x)$.

Hence $f(y)-f(x)<f'(x)(y-x)$.

The case $x>y$ is similar.

2(a) The case $x=0$ is obvious. Consider $x>0$. Let $f(x)=ln(1+x)$.

By Taylor's theorem, $\exists c\in(0,x)$ s.t.

$f(x)=f(0)+f'(0)x+\dfrac{f^{(2)}(0)}{2!}x^2+\dfrac{f^{(3)}(c)}{3!}x^3$.

$\Rightarrow ln(1+x)-x+\dfrac{x^2}{2}=\dfrac{1}{(1+c)^3}\dfrac{x^3}{3}$.

Since $0<\dfrac{1}{(1+c)^3}<1$ for $c>0$, the inequality follows.

2(b) By 2(a), $\dfrac{-\frac{x^2}{2}}{xsinx}\le\dfrac{ln(1+x)-x}{xsinx}\le\dfrac{\frac{x^3}{3}-\frac{x^2}{2}}{xsinx}, \forall x\in(0,\pi)$.

Since $\lim_{x\to0^+}\dfrac{-\frac{x^2}{2}}{xsinx}=-\dfrac{1}{2}=\lim_{x\to0^+}\dfrac{\frac{x^3}{3}-\frac{x^2}{2}}{xsinx}$, by squeeze theorem $\lim_{x\to0^+}\dfrac{ln(1+x)-x}{xsinx}=-\dfrac{1}{2}$.

Alternative solution:

Apply l'Hôpital's rule twice.

3(a) f is differentiable at c hence continuous at c, so $\lim_{x\to c}(f(x)-f(c)-f'(c)(x-c))=0$.
Also, $\lim_{x\to c}(x-c)^2=0$ and $f(x)-f(c)-f'(c)(x-c)$, $(x-c)^2$ are differentiable on I.
By l'Hôpital's rule, $\lim_{x\to c}\dfrac{f(x)-f(c)-f'(c)(x-c)}{(x-c)^2}=\lim_{x\to c}\dfrac{f'(x)-f'(c)}{2(x-c)}=\dfrac{1}{2}f''(c)$.
The last equality follows from the condition that f' is differentiable at c.

3(b) Define $\phi(x)=\begin{cases}\dfrac{f(x)-f(c)-f'(c)(x-c)}{(x-c)^2} & \text{if } x\in I\backslash\{c\}\\ \dfrac{1}{2}f''(c) & \text{if } x=c\end{cases}$
It can be easily checked that $f(x)=f(c)+f'(c)(x-c)+\phi(x)(x-c)^2, \forall x\in I$.
By 3(a), $\lim_{x\to c}\phi(x)=\phi(c)$ hence ϕ is continuous at c.

4 For each $n\in\mathbb{Z}$, mean value theorem implies that $\exists x_n\in(n,n+1)$ such that $f'(x_n)=\dfrac{f(n+1)-f(n)}{n+1-n}\Rightarrow|f'(x_n)|\le|f(n+1)|+|f(n)|\le 2$.
Claim: $|f'(x)|\le 3, \forall x\in\mathbb{R}$.
Let $x\in\mathbb{R}$. If $x=x_n$ for some $n\in\mathbb{Z}$, done.
Otherwise, $x\in[n,n+1]$ for some $n\in\mathbb{Z}$ and mean value theorem implies that
$\exists c$ between x and x_n such that $f''(c)=\dfrac{f'(x)-f'(x_n)}{x-x_n}$.
$\Rightarrow |f'(x)|=|f'(x_n)+f''(c)(x-x_n)|\le|f'(x_n)|+|f''(c)||x-x_n|\le 2+1\cdot 1=3$.

Alternative solution:
Claim: $|f'(x)|\le\frac{5}{2}, \forall x\in\mathbb{R}$.
Fix $x\in\mathbb{R}$. By Taylor's theorem, $\exists c\in(x,x+1)$ such that
$f(x+1)=f(x)+f'(x)(x+1-x)+\dfrac{f''(c)}{2}(x+1-x)^2$.
$\Rightarrow|f'(x)|\le|f(x+1)|+|f(x)|+\frac{1}{2}|f''(c)|\le 1+1+\frac{1}{2}\cdot 1=\frac{5}{2}$.